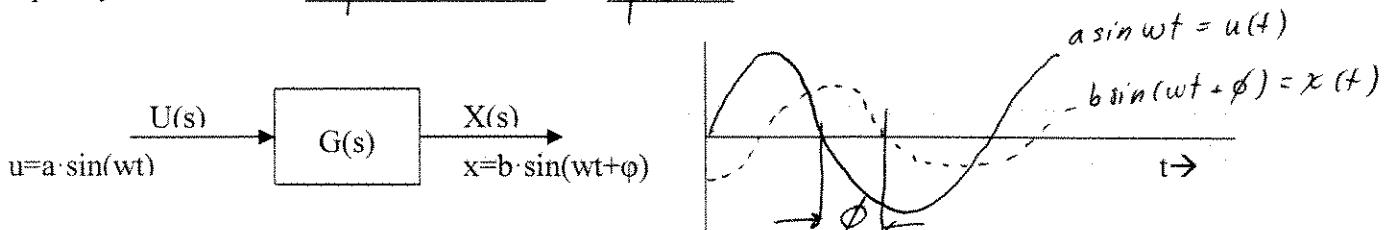


Mechanical Systems Laboratory

Frequency Response: Sine wave in, sine wave out with different amplitude and phase

Overview:

If we input a sine wave to a stable linear system, the output will be a sine wave of the same frequency with different amplitude and phase.



For the signals shown $x(t)$ lags $u(t)$ by ϕ , and this means that $\phi < 0$.

Mathematical model of the system:

How can we predict the output amplitude and phase? Consider a general n^{th} order linear system:

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_0 x = b_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + b_{n-2} \frac{d^{n-2} u}{dt^{n-2}} + \dots + b_0 u$$

If we take the Laplace Transform:

$$X(s^n + a_{n-1}s^{n-1} + \dots + a_0) = U(b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0) + IC(s) \rightarrow X = \frac{B(s)}{A(s)} U + \frac{IC(s)}{A(s)}$$

$A(s)$ is called the characteristic polynomial of the system, and we can write it in factored form:

$$A(s) = (s - s_1)(s - s_2) \dots (s - s_n)$$

What happen if $U(s)=0$?

$$X = \frac{IC(s)}{A(s)} \rightarrow X = \frac{C_1}{s - s_1} + \frac{C_2}{s - s_2} + \dots + \frac{C_n}{s - s_n} \xrightarrow{\mathcal{L}^{-1}} x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} + \dots + C_n e^{s_n t}$$

The system is stable if $\operatorname{Re}(s_i) \leq 0$, where s_i are the roots of the characteristic polynomial $A(s)$.

Assume we have a stable system, and we apply a sine wave to the system:

$$u(t) = a \cdot \sin wt \quad \mathcal{L} \rightarrow U(s) = \frac{aw}{s^2 + w^2}$$

$$X(s) = \frac{B(s)}{A(s)} \frac{aw}{s^2 + w^2} + \frac{IC(s)}{A(s)} \xrightarrow{\text{since } \operatorname{Re}(s_i) < 0 \rightarrow \text{decays to zero}}$$

Since $A(s)$ is stable, the contribution of the response due to the initial conditions decays to zero. Use partial fraction expansion to find the inverse Laplace transform.

$$X(s) = \frac{B(s)}{A(s)} \frac{aw}{s^2 + w^2} = \underbrace{\frac{k_1}{s + jw} + \frac{k_2}{s - jw}}_{\text{roots of } s^2 + w^2} + \underbrace{\frac{k_3}{s - s_1} + \frac{k_4}{s - s_2} + \dots}_{\text{roots of } A(s)}$$

The inverse Laplace transform has the form: (Trick: $L[e^{-at}] = \frac{1}{s+a}$)

$$x(t) = k_1 e^{-jwt} + k_2 e^{jwt} + k_3 e^{s_1 t} + k_4 e^{s_2 t} + \dots$$

because $\operatorname{Re}(s_i) < 0$

The terms e^{st} decay to zero, because $\operatorname{Re}(s_i) < 0$. Use partial fraction expansion to find K_1 and K_2 .

$$(s+j\omega) \frac{B(s)}{A(s)} \frac{aw}{s^2 + \omega^2} = \frac{k_1(s+j\omega)}{s+j\omega} + \frac{k_2(s+j\omega)}{s-j\omega} \Big|_{s=-j\omega} \rightarrow k_1 = \frac{B(s)}{A(s)} \frac{aw}{s^2 + \omega^2} (s+j\omega) \Big|_{s=-j\omega} \Rightarrow$$

$$k_1 = \frac{B(s)}{A(s)} \frac{aw}{(s+j\omega)(s-j\omega)} \Big|_{s=-j\omega} = \frac{B(-j\omega)}{A(-j\omega)} \frac{aw}{(-2j\omega)} = -\frac{a}{2j} G(-j\omega)$$

$$\text{Similarly, } k_2 = \frac{a}{2j} G(j\omega)$$

We can write $G(j\omega)$ in polar coordinates as: $G(j\omega) = |G(j\omega)| e^{j\varphi}$, where $\varphi = \angle G(j\omega)$. So that we can express $G(-j\omega)$ as: (hint: $G(-j\omega)$ is the complex conjugate of $G(j\omega)$). $G(-j\omega) = |G(j\omega)| e^{-j\varphi}$

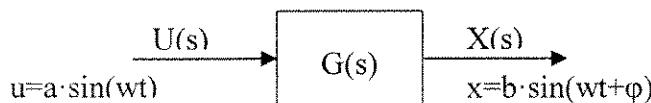
$$\begin{aligned} x(t) &= k_1 e^{-j\omega t} + k_2 e^{j\omega t} = -\frac{a}{2j} G(-j\omega) e^{-j\omega t} + \frac{a}{2j} G(j\omega) e^{j\omega t} = \\ &= -\frac{a}{2j} |G(j\omega)| e^{-j\omega t} + \frac{a}{2j} |G(j\omega)| e^{j\omega t} = \frac{a}{2j} |G(j\omega)| (e^{j\omega t + \varphi} - e^{-j\omega t - \varphi}) \end{aligned}$$

But using Euler's formula: $\sin(\theta) = \frac{1}{2j} (e^{j\theta} - e^{-j\theta})$, we get an output of the form:

$$x(t) = a |G(j\omega)| \sin(\omega t + \varphi)$$

Summary:

If we input a sine wave of amplitude a to a stable linear system, the output will be a sine wave of the same frequency with different amplitude and phase.



The output amplitude b can be computed as:

$$b = a \cdot |G(j\omega)|$$

The output phase can be computed as:

$$\varphi = \angle G(j\omega)$$

Example:



The output amplitude b can be computed as:

$$b = a \cdot |G(j\omega)| = a \frac{1}{\sqrt{1+(z\omega)^2}}$$

The output phase can be computed as:

$$\varphi = \angle G(j\omega) = 0 - \tan^{-1}\left(\frac{z\omega}{1}\right)$$